Evaluación de una estrategia didáctica para la apropiación del concepto "derivada de una función"

Evaluation of a teaching strategy for the appropriation of the concept: "derivative of a function"

Avaliação de uma estratégia de ensino para a apropriação do termo "derivada de uma função"

Felipe Santoyo Telles
Universidad de Guadalajara, México
felipes@cusur.udg.mx

Miguel Ángel Rangel Romero
Universidad de Guadalajara, México
marangel@cusur.udg.mx

Eliseo Santoyo Teyes
Universidad de Guadalajara, México
esantoyo25@hotmail.com

Karla Liliana Puga Nathal
Universidad de Guadalajara, México
karlalpn4@hotmail.com

Resumen

La enseñanza tradicional del cálculo diferencial presenta grandes dificultades respecto a la apropiación significativa del concepto *derivada de una función*. Dado lo anterior, se construyó una propuesta didáctica considerando al conocimiento como una construcción personal a partir de los esquemas de cada sujeto y una negociación intersubjetiva de significados; asimismo, se considera al profesor como un promotor o mediador de la interacción entre los sujetos cognoscentes y el objeto cognoscible. Sobresalen los procesos de organización y adaptación con una estructura que atiende los principios del proceso de asimilación–acomodación de Piaget, y que provocan el cambio de estructura, desarrollo y aprendizaje. Los contenidos de la propuesta se presentan contextualizados y organizados bajo una secuencia lógica en un cuadernillo de trabajo. Finalmente, la propuesta se experimentó y valoró mediante la prueba *t student* con resultados positivos.

Palabras clave: estrategias didácticas, aprendizaje significativo, derivada de una función.

Abstract

The traditional teaching of the Differential Calculus presents great difficulties concerning the significant appropriation of the concept *derivative of a function*. Given it above, is built a proposed didactic whereas to the knowledge as a construction staff with base in the schemes of each subject and an intersubjective negotiation of meanings; also, the Professor is considered as a promoter or mediator of the interaction between the cognoscenti subjects and the knowable object. Stand out those processes of organization and adaptation with a structure serving the principles of the process of assimilation-accommodation of Piaget, and causing the change of structure, development and learning. The contents of the proposal are contextualized and organized under a logical sequence in a work booklet. Finally, the proposal is experienced and valued by the test *t student* with positive results.

Key words: educational strategies, meaningful learning, derivative of a function.

Resumo

O ensino tradicional de cálculo diferencial apresenta grandes dificuldades quanto à apropriação significativa do derivado conceito de uma função. Face ao exposto, a proposta didática considerando o conhecimento como uma construção pessoal dos esquemas de cada sujeito e uma inter significados de negociação construídos; Além disso, é considerado o professor como um promotor ou mediador da interação entre os sujeitos cognitivos e objeto cognoscível. Projetam processos organizacionais e adaptação com uma estrutura que serve os princípios do processo de assimilação-acomodação de Piaget, e causar a mudança de estrutura, desenvolvimento e aprendizagem. O conteúdo da proposta são contextualizadas e organizada de acordo com uma sequência lógica de um livro. Finalmente, a proposta foi experimentado e apreciado pelo teste t de Student com resultados positivos.

Palavras-chave: estratégias de ensino, aprendizagem significativa, derivada de uma função.

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Introduction

Talk about the teaching of the mathematics in Mexico leads us to a road that has been difficult and complex, imbued with both personal and institutional obstacles and failures. Esto se puede observar a partir de los altos índices de reprobación in the Basic levels, College and Higher education, as well as in the reports of the Programme for International Student Assessment (PISA), o los de la National Assessment of Educational Progress (ENLACE by its name in Spanish), among others.

This problematic in the teaching of the mathematics has been documented since several decades ago; the master programs of common trunk subjects of the technical high school (Secretariat of Public Education, 1988) pointed to a series of failures in the teaching of mathematics in general and mentioned that the traditional teaching of Differential Calculus presents large difficulties concerning the significant appropriation from the students side of the concept derivative of a function. On this regard, Artigue (1995, p. 97) notes:

> Is clear that the teaching of Calculus principles is problematic. Numerous investigations performed show, with surprising convergences, that while it can teach to the students to perform in a way more or less mechanical some derivative and primitives calculations and to solve some standard problems, are found great difficulties to make them truly enter in the field of Calculus and to make them reach a satisfactory understanding of the concepts and methods of thinking that are the center of this part of mathematics.

Similarly, Moreno (2005) reiterates that the teaching of Calculus is fairly problematic. Although we are capable of teach to the students to perform some Derivatives, such actions are far away from a true understanding of the concepts and methods of thinking of this part of mathematics. Ávila (1998, p. 1) says about the teaching of the calculation that “not has been possible design a strategy that ensures that the students acquire the right level of conceptual and methodological domain to succeed in the typical problems solving of the discipline”.

Referring to the teaching of differential calculus, Sánchez Matamoros et al., (2008) mention multiple research on teaching and learning of variational and differential and integral calculus, noting that such investigations coupled with experience as teachers calculation they have allowed them to check the difficulty of teaching and learning such concepts. For his part, Zúñiga (2007) mentions that this situation has been addressed in multiple jobs (Farfan, 1991 and 1994; Artigue, 1995, Dolores., 1999; and Salinas et al, 2002) showing theoretical arguments and proposals to improve the quality of learning in teaching calculus.
From these and other studies, the Directorate General of Industrial Technology Education (DGETI), drove in two major reforms (2004 and 2008), which permeated to the upper levels across the country. However, five years into the Reform of School Education (RIEMSER 2008) shows that both the CBTIS 226 (CBTis 226) and in the University Center South (CUSUR) University of Guadalajara (University of Guadalajara), students fail to build and own the concept of derivative of a function, and apply that knowledge to solve various problems, among others, optimization; sometimes students mathematical algorithms address the bypass as a collection of very complex rules (at times meaningless) they have to memorize and reproduce to pass the corresponding learning unit.

Use averages obtained at the departmental calculus test in CUSUR over two semesters - Cycles 2013A- 2012B and were respectively 35 and 48 with a failure rate of 56% and 64%.

It also notes that often both textbooks and teachers, begin with the definition of a concept to continue with a series of examples, keeping students as spectators in the repetition of a series of algorithms. In addition to this, there are many history teachers in middle and higher levels whose students do not have the background needed to understand the derivative of a function.

In this paper the results of implementing a teaching strategy based on constructivism, which began with the design of a number of optimization problems and continued with their resolution by the students are presented. With her understanding and appropriation of the concept of a function derived through problem situations in the area of professional interest of students are promoted.

To address mentioned this analysis was divided into three sections: in the first part the theoretical foundations that support the educational proposal, in the second part the methodological design that followed appears to contrast the variables and to implement present the proposal in an experimental group of 45 students of technical high school, and the third evaluation of the results and its contrast with a group (control) of students of the same institution where the strategy has not worked is presented. For practical purposes, the teaching strategy that was designed and implemented is presented in the section of annexes.

General Objective: To determine the effect of the teaching strategy designed for learning the concept of derivative of a function.

Hypothesis: there are significant differences in the appropriation of the concept derivative of a function, to implement the teaching strategy designed regarding the appropriation achieved with the traditional way of teaching differential calculus.

Literature review

Knowledge is always a transitional state of a process; know is to assimilate and not copy. Assimilate, according to Glasersfeld (1997), means above all interpret, give meaning to a new experience built from the subject's cognitive schemata. In this section we present some of the basic concepts related to the construction of knowledge from meaningful learning.

The contents in a problem situation

A cognitive schema can be a concept or a pattern of action; in a scheme there is always a recognition mechanism and some expectation about the results expected by the activation of the scheme. If the results are consistent with those expected, then the scheme becomes more stable and reliable; however, it may be that facing a new situation a scheme does not respond properly, then the cognitive schema is unbalanced and there is the need to respond to the disturbance. This is achieved by modifying the scheme in question, ie the accommodation, resulting in the re-equilibration system. Thus, it appears that learning is the consolidation of cognitive schemata (patterns of action, concepts, theories, etc.) and the generation of new ones from existing imbalances, once these are insufficient to tackle new tasks .

One of the variables that influence learning-particularly those of mathematical concepts in character is the way they are presented by the teacher in the context of school learning, where these concepts are structured and applied specifically as part design of teaching discipline in the classroom knowledge. This process Chevallard (1996) describes it as "didactic transposition".

For example, during the teaching of differential calculus an essential concept as "the derivative of a function" can be understood by students only as an "operation" if the activity in the classroom is focused on the manipulation of symbols through rules and formulas. Teaching such incident in a weak construction of meanings of the concepts.

In order to understand more deeply the above concept, students appropriating its meaning as instantaneous rate of change or slope of a line tangent to the curve at a given point, which also represents an instantaneous rate of increase or decrease phenomena whose variation follows a pattern given by the function.

So to encourage students to take ownership of the concept derivative of a function, it is recognized that should promote meaningful learning from problem solving situations. Zuniga (2007) agrees with the idea that the mathematical study of both real-world phenomena as mathematical, place the learner to problem situations. In this regard, Douady (1993, p.5) states:

> A student has knowledge of mathematics if he is able to bring its operation as explicit tools on the problems to be solved, whether or not indicators in the formulation, and is able to adapt when the usual conditions of employment are not exactly satisfied, to interpret problems or raise questions in their regard.

In this sense, Buendia and Ordoñez (2009) point base construction significance to concepts of calculus and pre-calculus from the development of own thought and variational language strategies, particularly for the significant appropriation of the derivative of a function. In this regard, they mention that study what varies, while how in changing phenomena can provide the meanings derived from management away derivation formulas, something which is usually limited his teaching.

Meanwhile, Cantoral and Farfán (1998) point out simultaneous and coordinated management of the successive derivatives as a condition without which the formation of the idea of the derivative becomes fragile; however, this paper addresses the relationship between two quantities that vary and are functionally related and where the variation of one depends on the other, from what Camarena (2000, p. 23) names mathematics in context. In this regard, he states:

> Mathematics in context helps students build their own knowledge of a mathematical meaningful, with firm and non-volatile moorings; also it reinforces the development of mathematical skills through the process of solving problems related to the interests of the student.

Zúñiga (2007) It describes the characteristics of each stage in processing information Feuerstein cognitive functions that appear in the mental act of learning involved in solving a problem. The author identifies three stages in the appropriation of a concept from the resolution of a problem situation:

- *Phase input.* Understanding implies for the student clearly understand both the data offered in the initial information, such as the final state or goal that you want to reach. To achieve clear perception is necessary for the cognitive functions of systematic exploration of a learning situation, arising efficiently.
- *Processing Step.* It involves the search for alternative solutions that connect the initial state with the goal to achieve (the resolution of the problem situation), but it is necessary before the subject is able to perceive and accurately define the problem, which means that its function cognitive perception and definition of this arises efficiently. This moment is the link between the understanding of the problem situation and what is properly solving the problem.
- *Output phase.* The answer must be issued using clear and precise language depending on the ultimate goal of the formulated problem, that is, it should be noted explicit communication of such a response (pp. 153-154).

The need to promote deep reflection on the concept and not merely as an instrumental tool treatment is also recognized, allowing students to be able to apply such a concept to solve problems of interest. With regard to the above, Godino and Recio (1998) suggest that the meaning is clear from the actions that the student runs on mathematical objects, calling them "significant prototypical practices".

It is also known that in most mathematical concepts may be involved different domains or frames of representation: physical, geometrical, numerical, graphic (Douady, 1993); further, to reach a deep and lasting understanding is advisable to promote manipulation of mathematical object from different frames of representation.

In genetic epistemology developed by Piaget, according to various authors as Woolfolk (1996), Moreno (1998), Garcia (2006) and Serulnicov (2010), learning takes place from two fundamental principles: "assimilation and accommodation." The assimilation of an object or situation involves an interpretation, which is necessary to make admissible the object to be processed by the cognitive structure (accommodation), the result of this process is a form of knowledge that is not the result of " copy "the external data (reality) as presents to the senses. The fundamental assumption is that humans construct, through experience, their own knowledge and not simply receive the processed information to understand and use immediately. Knowledge is the result of a continuing construction from the world of our experiences.

With the work of Piaget it has become clear that constructivism allows the development of the ability to make meaningful learning in creating "learning situations" that emphasize reflective learner activity. Just this position corresponds to the proposed strategy, creating learning situations with a clear intention and accurate; it is recreating a problem condition for students to interact with the object of knowledge and in the process take ownership of it.

According to the ideas expressed by Glasersfeld (1997, p. 2), "both biologists and physicists recognize that the conceptual structures that we consider as knowledge, are the products of active knowers who shape their thinking to fit the constraints they experience ". Then it comes to confronting the student to a problem situation to promote step by step reconstruction of a concept, which is held in a previous scaffolding, so it is necessary to ensure an understanding of some basic concepts and mastery of certain basic skills that enable students by building the concept derivative of a function.

Teaching strategy

According to Piaget et al. (1977), the problem statement and its specific structure allows cognitive conflicts generated the student try to resolve from the schemes has, then assimilate; that is, try to solve the problem using the knowledge and resources it has, and if necessary their cognitive scheme will be modified to accommodate the situation he faced. The actions promoted (steps to resolve the problem) have the subject in two directions: to reaffirm something you already know or restructure their schemes and give meaning to a new experience, so that knowledge is both time consuming, result, the starting point and process.

In other words, in the scheme they are always present a mechanism of recognition and a certain expectation about the results expected by the activation of the scheme. If the results (after action on the problem) are compatible with the expected, then the scheme becomes more stable, but it can happen to face a new situation (problem or part of the problem) a scheme does not respond properly then cognitive scheme is unbalanced (this is precisely the proposal, unbalancing the system from a problem situation that must be resolved), and the need arises to answer (solve) to the disturbance. This is achieved by modifying the scheme in question, this is the accommodation.

Cognitive experience of human beings is not confined to their interaction with material things, but includes the results of the interaction with other people in their social environment. In the process of solving optimization problems it is proposed that students form teams of three elements and develop a collaborative work, since according to Moreno (1998, p. 169), "the interaction with others -especially when ways of acting and the means available for such action been insufficient listed as one of the main sources of cognitive imbalance and therefore learning. "

Methodology

We worked with a procedural methodology based on problem-solving optimization as a core activity. Also, special care that problem situations will have a logical structure, establishing links between prior knowledge structures and purchase new content to be had. The willingness of students to learn is important, so the problems are related to their immediate reality, considering that if they find meaning and practical application will then be more attractive.

Kind of investigation. Quantitative cross-cut by applying a test.

Instruments. The test consists of eight open questions that require students to manifest understanding of the concepts necessary for the construction of the concept of derivative of a function. Once the responses were obtained, they were evaluated by contrasting with the meanings institutionalized in various textbooks, and the test result was rated from zero to 100 points. On the other hand, it was established by statistical analysis the level of significance between the results obtained by working with two groups, one experimental "A" and a control group "B".

Universe. The universe was made up of the school population attended the fourth semester CBTIS of Ciudad Guzman, Jalisco, Mexico (CBTis No. 226).

Sample. They were randomly assigned to two groups, which were determined by the head of the department of educational services of the institution. The experiment was conducted during the months of February, March, April and May, and two focus groups were treated with different specialties; the group working with the proposed methodology was called experimental "A" (44 students) and who worked with the traditional methodology was called control "B" (46 students). In the experimental group randomly teams were formed, students from each team remain in it during the course of the experiment to achieve integration and to facilitate the negotiation of new meanings.

Statistical analysis (application of the proposal)

The results of this study were analyzed using SPSS version 15 (Chicago, IL, USA). student t-test was used at a level of significance to corroborate the existence of statistical differences $p \leq 0.05$.

Results

The averages obtained as a result of the application of a test made with the intention that the student expresses understanding of the concepts necessary for the construction of the concept of derivative of a function were compared. The group worked with the experimental methodology obtained an average of 60.29 ± a standard error of 3.8 points, while the group that worked with the traditional methodology (control) the average was 24.12 with a typical error of 3.41. The results are shown in the following table.

Grupo	N	Media	Desviación típica	Error típico de la media
Control	39	24.1282	21.3112	3.41252
Experimental	44	60.2955	25.40608	3.83011

Table 1. Descriptive values

Statistically significant differences in the averages obtained between the control group and the experimental group ($p = 0.000$) were found. The difference between the sample means (see figure below), is not the result of sampling error. It can be said that under the conditions in which the study was conducted, the methodology was crucial for students to appropriate the concept of derivative of a function.

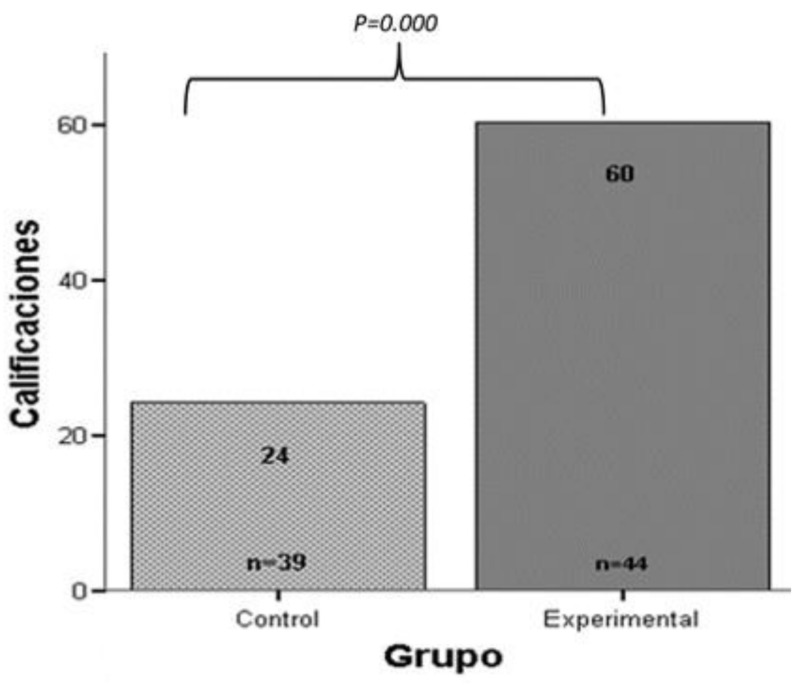

Figure 1. Statistical significance (elaboración propia).

Global response to the problem

From the results obtained in this investigation it can be said that using the methodology proposed in the experimental group, students in the fourth semester of technological baccalaureate appropriate the concept of derivative of a function and develop the appropriate skills for problem solving optimization involving differential calculus.

Discussion

It was found enough evidence to say that through the teaching strategy that was used in the experimental group students:

- They appropriate the necessary elements that allow them to develop and interpret records representing various functions.
- Reconstruct a process through which they can develop and internalize the concept of derivative and appropriating it, and understand and apply the knowledge gained in solving optimization problems.
- Develop strategies teamwork and leadership.

According to the observations and records made, it can be said that the teaching strategy based on solving optimization problems is an important option to encourage students to achieve meaningful learning of the derivative of a function, and simultaneously promotes internalization process of orderly and systematic (organization and structure) that enhance the activity of the students, their intrinsic motivation and about the lasting acquisition of skills and attitudes that prevail in the continuum of their academic and professional preparation thought.

To complement the reported data, anticipate that students showed skills such as note taking, rescue important, relate concepts, organizing time, verbally describe what you do, understand before solving, drawing, using model problems, changing frames representation, explore the problem, be flexible in the process, learn from the mistake, placed in context, identify, outline, discover relationships, all of which although not developed from the work with the teaching strategy, yes invites inquiry detailed.

Contrast basics with the main results of the work

- When mathematize situations of everyday life as a process for working reality through ideas and mathematical concepts, such work was carried out in two opposite directions: from the context created schemes formulated and visualized the problems, relationships and regularities were discovered, similarities with other problems were found. And by working mathematically they found solutions and proposals that were projected on reality to analyze its value and meaning.
- The structure of the problems solved favors students step through the stages, ranging from reading the statement to the formulation of the response, including the design and implementation of a plan that proved to be guiding the suggested activity. In this context the interaction of arithmetic, geometric and algebraic frameworks favored.
- Significant learning were achieved, as students established links between residents prior knowledge in cognitive structure and new content embodied in an orderly way the problems resolved.
- Inductively learning were proposed and the teacher remained a guiding attitude oriented allowing the simple to the complex discovery.

- The proposed strategy allowed to maintain a close relationship between learning and cognitive development; students, interacting with reality, cognitive conflicts generated that favored the process of self-structuring.
- When solving problems in teams, the social dimension of learning, which is an inter negotiating meanings that are built on the interaction of students with their peers and the teacher was considered. The latter is essentially manifested as promoter of the interaction between cognitive subjects and knowable objects.

Conclusion

Quantitative analysis showed positive responses regarding the appropriation of the concept of derivative of a function. However, it is important to note that there are multiple relationships and explore aspects from the qualitative approach; for example, the values internalized by the students, the degree of significance of the concepts, leadership skills, among others. In addition, it is known that knowledge is always partial, temporary and perfectible, a continuous process under construction, and also that each group and each student is different. From these premises may be mentioned the following limitations:

- Resistance to change by some students who prefer the teacher to explain them, while they merely take notes and follow instructions.
- Difficulty to develop teamwork.
- The proposed work performed in a given context, but is not sure that works the same in other contexts.
- In some cases the generalization from the inductive process is not given in full, so some students reach not see beyond the particular situation treated.
- For some students the problems were not of his personal interest or professional.
- Suspensions official and unofficial class (outside scheduled at the beginning of each semester) negatively altered the course planning.
- Tardiness or absence of some pupils, affecting the planned development process.

The proposed scenarios are focused on problems of economic-administrative area, however, they can be designed similar situations to use this methodology in almost any other area, for example, in programs of medical-biological area or in the areas of engineering, mechanics, physics, etc. In general, students in the experimental group achieved a greater understanding of the concepts, develop better procedures and a considerable improvement in their attitude towards study and work habits, which can be studied better from the qualitative approach.

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Anexo

Por cuestiones de espacio se describe solo uno de los cinco problemas elaborados en la propuesta.

Problema

El responsable de la cafetería del plantel estima un costo de elaboración por cada desayuno que prepara en $\$10.^{00}$, también estima sus costos fijos por conceptos de renta, energía eléctrica, impuestos, empleados, etcétera, en $\$300.^{00}$ diarios. El encargado de la caja ha observado que vendiendo cada desayuno en $\$16.^{00}$ la demanda es de 140 desayunos, pero cuando el precio sube hasta $\$22.^{00}$ la demanda baja a 80 desayunos.

¿Cuánto será la mayor ganancia posible a lograr?, ¿cuál será el mejor precio de venta, de tal manera que le permita obtener las mayores ganancias?, ¿cuántos desayunos venderá a ese precio? y ¿a partir de un detallado análisis qué recomendaciones o comentarios se podrían hacer?

Primera parte

1.- Lee cuidadosamente y comprende el problema. Separa los valores conocidos y determina cuál es la incógnita, ¡escríbela!

2.- Necesitamos un plan para resolver este problema. Construiremos la función de utilidades, (en su expresión algebraica). Relaciona "x" tortas vendidas con "U" utilidades, esta es la parte principal del plan que nos permitirá saber cuándo obtendremos mayores utilidades. Te guiaremos en este camino.

3.- A partir de la información dada establece la función de costos por día (a cuánto ascienden sus gastos o costos por día) y grafícala. **C = m x + b**. Donde "**m**" es el costo de elaborar cada desayuno y "**b**" es el costo fijo.

4.- Con la información dada establece la expresión algebraica (función) que relacione precio "p" y ventas "x" y traza la gráfica correspondiente. Nota: considera en esta ocasión que por conveniencia el precio depende del número de desayunos vendidos. Puedes plantear la expresión del siguiente modo: $P - P_1 = m\,(x - x_1)$ en donde $m = \frac{p_2 - p_1}{x_2 - x_1}$ "P" estará en función de "x".

5.- Con las dos ecuaciones anteriores establece la función de utilidades considerando que las utilidades son ingresos menos costos. **U = I – C** (Ingresos = No. de desayunos "x" por "P" precio de cada desayuno). **U = I – C = x P – C.** Atención, "P" precio lo tomas de la segunda función paso 4 y "C" costo de la primera función paso 3.

Desarrollando el plan

6.- Traza la gráfica correspondiente de la función de utilidades, ubicando en el eje de las abscisas la variable "x" y en el eje de las ordenadas la variable "U". Observa que "U" es función de "x", o sea U depende de x, dicho de otro modo *U=f(x).*

7.- En el punto (20, 60), ¿la curva de *U*= "***f (x)***"está creciendo o decreciendo?

8.- En el punto (110, 690), ¿la curva ***f (x)*** está creciendo o decreciendo?

9.- Traza una recta tangente a la curva de f(x) con la condición de que la pendiente de esta recta tangente sea cero.

10.- Escribe cual es el punto en la curva en el que tendrás máximas utilidades. Para hallar este punto, ¿qué tuviste que hacer?

11.- Traza una recta tangente a la curva por un punto cualquiera. ¿Puedes determinar qué dirección tiene la curva en ese punto?, ¿conoces algún método para determinar esta dirección?

12.- ¿Para qué crees que te sirve en este caso conocer la dirección de la curva en un punto en particular?

13.- ¿Qué importancia podría tener conocer la pendiente de la curva en algún punto en particular?

14.- Toma el punto (x_1=20, U_1=60) de la función de utilidades ***y = f (x),*** luego asigna un incremento a "x_1" ("Δx") de una unidad $x_2 = x_1+\Delta x = 21$ y calcula el respectivo valor de $U_2 = f(x_1 + \Delta x)$. Observa que también $U_2 = U_1 + \Delta U$. Observa que en este punto al aumentar x, también aumenta U.

15.- Como obtendrás dos puntos (x_1, u_1) y (x_2, u_2) traza una "recta secante" que pase por esos dos puntos y calcula su pendiente $m = \frac{U_2 - U_1}{x_2 - x_1}$

16.- Manteniendo fijo el punto (x_1, y_1) ahora incrementa a "x" "Δx", de tal modo que el incremento sea más pequeño que el anterior. Por ejemplo, Δx = ½ calcula nuevamente el respectivo valor de U_2, calcula también ΔU y la pendiente de la recta secante que pasa por estos dos puntos (x_1, U_1) y el nuevo (x_2, U_2).

17.- Repite el paso anterior algunas veces más, ayúdate con la siguiente tabla. Utiliza la función de utilidad que determinaste en el paso número 5.

x_1	x_2 $(x_1+\Delta x)$	$\Delta x = x_2 - x_1$	U_1	U_2	$\Delta U = U_2 - U_1$	$m = \Delta U / \Delta x$
20	21	1	60	75.9	15.9	15.9
20	20.5	½ = 0.5	60			
20	20.2	$\frac{1}{5} = 0.2$	60			
20	20.1	$\frac{1}{10} = 0.1$	60			
20	20.01	$\frac{1}{100} = 0.01$	60			

18.- A partir de la tabla anterior se promueve que el alumno vaya haciendo los incrementos de la variable independiente cada vez más pequeños, hasta tender a cero. Al final se le pregunta que si de modo intuitivo podemos concluir que cuando el incremento "Δx" **tiende** a ser cero, ¿qué valor tendrá la pendiente $m = \Delta u / \Delta x$? Si el alumno realizó la serie de pasos adecuadamente, ¡¡Felicidades, ha calculado de modo intuitivo aritmético y puntual la (derivada) pendiente de la recta tangente a la curva de la función de utilidades en el punto (20, 60)!!

Segunda parte. Generalizando: la derivada en un plano algebraico

1.- Considera fijo el punto (25, 137.5) y completa la siguiente tabla igual que la anterior. Observa que:

- $f(x_1)$ corresponderá a U_1, por lo tanto $f(x_1 + \Delta x)$ corresponderá a U_2
- $f(x_1 + \Delta x) - f(x_1)$ es igual a ΔU, por lo tanto $\frac{f(x_1+\Delta x) - f(x_1)}{\Delta x}$ es igual a $m = \Delta u / \Delta x$
- Recuerda que nuestra función objetivo es $U = -\frac{x^2}{10} + 20x - 300$

x_1	$x_2 = (x_1+\Delta x)$	Δx	$f(x_1) = U_1$	$f(x_1+\Delta x) = U_2$	$f(x_1+\Delta x) - f(x_1)$	$\frac{f(x_1+\Delta x) - f(x_1)}{\Delta x} = m = \frac{\Delta U}{\Delta x}$
25	26	1	137.5	152.4	14.9	14.9
25	25.5	½ = 0.5				
25	25.2	$\frac{1}{5} = 0.2$				

25	25.1	$\frac{1}{10}=0.1$				
25	25.01	$\frac{1}{100}=0.01$				

2.- Dado que $m=\Delta u/\Delta x$ escribe el valor que tendrá la pendiente en el punto (25, 137.5) cuando el Δx tienda a cero.

¡¡Felicidades, ahora has calculado de modo aritmético la pendiente de la recta tangente a la curva de la función de utilidades en el punto (25, 137.5)!!

3.- Las pendientes en el punto (20, 60) y (25, 137.5), ¿fueron iguales o desiguales?, ¿qué significa esto?

Después reflexiona sobre el siguiente concepto:

Atención: la derivada de una función es la tangente del ángulo de inclinación de la curva en un punto, es decir, es la pendiente de la recta tangente en ese punto; esto nos permite saber la dirección de la curva en tal punto. <u>Geométricamente la derivada es la pendiente de la recta tangente a la curva en un punto dado</u>.

Proceso algebraico para encontrar la pendiente de una recta tangente a la curva en cualquier punto

A continuación encontraremos la derivada "algebraicamente" de nuestra función de utilidades en estudio. $U_1=-\frac{x^2}{10}+20x-300$ Observa que: *f(x)* es U_1

f(x+ Δx) corresponde a $U_2 = -0.1(x+\Delta x)^2+20(x+\Delta x)-300$

- *f(x+ Δx) – f(x)* es igual a $U_2 - U_1 = \Delta U$ (se deja al alumno realizar la resta)
- ΔU $=-0.2x\Delta x-0.1\Delta x^2+20\Delta x$ si se divide esta expresión entre el incremento de la variable independiente Δx, entonces $\frac{\Delta U}{\Delta x}=\frac{f(x_1+\Delta x)-f(x_1)}{\Delta x}$ = $-0.2x-0.1\Delta x+20$
- Si el Δx tiende a cero, entonces $\frac{f(x_1+\Delta x)-f(x_1)}{\Delta x}$ = $\lim_{\Delta x\to 0}\Delta U/\Delta x=m=-0.2x+20$

La función obtenida g(x) $=-0.2x+20$ = $\lim_{\Delta x\to 0}\Delta U/\Delta x=m$ es la función derivada de la función $U=-0.1x^2+20x-300$ se le representa por $U'=-0.2x+20$

"esta función *derivada"* $U' = -0.2x + 20$ permite hallar el valor de la pendiente de toda recta tangente que se puede trazar en cualquier punto de la curva de utilidades, y conocer la dirección de tal curva. Si crece o decrece, tiene un valor máximo o uno mínimo.

Parte tres. Gráfica de la derivada (plano geométrico)

En esta sección se promueve que los alumnos logren una representación gráfica de la función y de la derivada de esta en un mismo plano y luego hagan el análisis correspondiente con algunas acciones como las siguientes:

1.- En el mismo plano que trazaste la gráfica de la función de utilidades, $U = -0.1x^2 + 20x - 300$, traza la gráfica de la función derivada $U' = -0.2x + 20$

2.- Compara las dos gráficas y responde a las siguientes preguntas:

- ¿Qué pasa en la gráfica de utilidades U cuándo la gráfica derivada U' es positiva?
- ¿Qué pasa cuando la derivada U' es negativa?
- ¿Qué pasa en la gráfica de utilidades cuando la gráfica derivada U' se hace cero?
- ¿Qué importancia tendrá el conocer el punto exacto donde la derivada se hace cero y cómo se podrá saber esto último?
- ¿Cómo interpretas tú personalmente la derivada?, ¿cómo lo expresarías con tus palabras?
- ¿Qué otras aplicaciones se te ocurren para la derivada?