Revista Iberoamericana para la Investigación y el Desarrollo Educativo
ISSN 2007 - 7467

https://doi.org/10.23913/ride.v13i25.1245

Artículos científicos

La didáctica del cálculo integral: el caso de los procedimientos de integración

The Didactic of Integral Calculus: The Case of Integration Procedures

A didática do cálculo integral: o caso dos procedimentos de integração

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Resumen

En el presente artículo se analizan algunos procedimientos adicionales para la solución de integrales que complementan la manera tradicional de enseñanza en el cálculo integral. Dado que el contenido de los libros de cálculo solo se enfoca en ciertos procedimientos, que se ven plasmados dentro del aula, se propone, en primer lugar, una ampliación del formulario básico de cálculo integral adicionando las funciones Lambert W y las funciones Leal recientemente publicadas. En segundo lugar, se propone una guía didáctica para la solución de integrales por partes. Además, en este trabajo se presentan el método de integración Hermite-Ostrogradsky y el método para integrales de funciones irracionales. Ambos métodos pueden considerarse como una extensión de la integración por fracciones parciales y de la sustitución trigonométrica por cambio de variable; así, se evita el discurso matemático escolar. Como objetivo, se exponen procedimientos facilitadores y novedosos en la didáctica del cálculo integral para estudiantes del nivel medio superior y superior.

Palabras clave: discurso matemático escolar, formulario de cálculo integral, funciones primitivas, integración por partes, técnicas de integración.

Abstract

This article presents some additional procedures in solving integrals to those generally presented in calculus books and in the classroom. In the first place, an extension of the basic integral calculus form is proposed by adding the Lambert W and trascendental Leal functions. Second, we propose a didactic solution guide for the solution of integrals by parts. In this work. The Hermite-Ostrogradsky integration methods and the German method for integrals of irrational functions are presented. Both methods can be considered as an extension of the integration by partial fractions and of the trigonometric substitution by change of variable, thus avoiding the school math speech.

Keywords: school math speech, integration formulas, primitive function, integration by parts, integration techniques.

Resumo

Este artigo analisa alguns procedimentos adicionais para a resolução de integrais que complementam a forma tradicional de ensino de cálculo integral. Dado que o conteúdo dos livros de cálculo se concentra apenas em determinados procedimentos, que são refletidos em sala de aula, propõe-se, em primeiro lugar, uma extensão da forma básica de cálculo integral adicionando as funções W de Lambert e o recém-publicado Leal funções. Em segundo lugar, é proposto um guia didático para a solução de integrais por partes. Além disso, neste trabalho são apresentados o método de integração de Hermite-Ostrogradsky e o método para integrais de funções irracionais. Ambos os métodos podem ser considerados como uma extensão da integração por frações parciais e substituição trigonométrica por mudança de variável; assim, evita-se o discurso matemático escolar. Como objetivo, são expostos procedimentos facilitadores e inovadores na didática do cálculo integral para alunos do ensino médio e superior.

Palavras-chave: discurso matemático escolar, forma de cálculo integral, funções primitivas, integração por partes, técnicas de integração.

Fecha Recepción: Noviembre 2021 **Fecha Aceptación:** Abril2022

Introduction

Although the school mathematical discourse (DME), also known as the mathematical language used in classes, has been present in the teaching of mathematics at different educational levels in Mexico, it has been the system of reason that has normally imposed arguments, meanings and procedures (Soto, Gómez, Silva and Cordero, 2012; Soto and Cantoral, 2014). Regarding the contents, these remain unchanged despite the reforms carried out in the educational system. This aspect is reflected in the teaching of the contents, since these remain practically the same (Cantoral, Montiel and Reyes, 2015). The same happens with the textbooks used for mathematics courses, since they are dominated by the DME (Cantoral et al., 2015).

Likewise, the DME is found in textbooks in different areas of engineering. For example, in the area of control systems (Ogata, 2003; D'Azzo and Houpis, 1995) the arguments for determining the root locus remain practically the same despite the fact that the authors published a new edition of the same book and publications that demonstrate other procedures that facilitate the teaching of integral calculus, as well as its learning in students. Although it is necessary to investigate theoretical aspects and traditional procedures, it is equally opportune to know and apply the new knowledge in the area.

The same case occurs with books on probability and statistics. For example, the way in which the authors approach the determination of the probability by means of some probability distribution does not change. Because they do it through the classic tables to determine the probability, which are made up of rows and columns that contain the probabilities that have been previously calculated and tabulated for consultation. This happens with the Gaussian probability function, also known as the normal distribution (Anderson, Sweeney, and Williams 2008; Daniel, 1987). It is worth mentioning that, in most cases, the tables are found in the appendices of the books (Anderson *et al.*, 2008; Daniel, 1987).

In relation to the calculation books, they are also conceived under the DME. The contents they address are the same, with the difference that some authors present more demonstrative examples in each of the topics, while others propose more exercises (Ayres & Mendelson, 1994; Purcell, Varberg & Rigdon, 2007). Likewise, some books present a better organization of contents, employ better didactics and even use one or more printing inks that help clarify the contents (Ayres and Mendelson, 1994; Purcell et al., 2007; Stewart, 2015; Stewart, Redlin and Watson, 2007).

However, in both engineering and mathematics disciplines, some authors have proposed different alternatives that can help overcome DME. For example, in Sandoval et al. (2019a) an algebraic expression was proposed for the solution of a circuit where the polarization of two rectifier diodes is shown. Traditionally, in analog electronics classes, this circuit is solved using graphical methods (Boylestad and Nashelsky, 2003).

Similarly, in Sandoval, Vazquez, Filobello and Hernandez (2019b) two approximations were proposed in terms of elementary functions, one for the error function and the other for the normal cumulative function, which simply substituted the numerical value of interest; in them it is possible to obtain the value of probability or of the error function. These numerical evaluation methods can help in statistics classes and other areas by showing alternative paths in their determination compared to the traditional tables where the DME is involved (Spiegel, 2009). The difference is that the result can be expressed as a function of the parameters of the problem to be solved, which can be a great advantage when some type of analysis is required by scanning variables.

Finally, Sandoval, Hernández, Torreblanca and Díaz (2021) presented an update to the propaedeutic contents of the disciplinary field of the technological baccalaureate of the General Directorate of Industrial Technological Education (DGETI) in Mexico, where it is proposed to include the teaching of the functions special, Lambert W and hyperbolic functions with some applications in order to bypass the DME.

Methodology

This work was carried out using a documentary research method, which consists of conducting searches about a particular topic with the aim of finding out what has been done on the topic of interest. This point was made in the best-known calculus books of various authors, since they are the means of support for teachers to teach calculus or related classes. In turn, books from different publishing houses in the English and Spanish languages were taken up with the aim of comparing the contents published in Mexico. Likewise, the updated editions were compared with previous editions in the books that have had updates. Regarding the books consulted, in the references of this article it has been considered to cite some books of authors most used in integral calculus courses at the high school level and science and engineering degrees. For example, in the calculus courses of the technological baccalaureate of the DGETI, these books are used as references, among others (Ministry of Public Education [SEP], September 4, 2012). This point allowed the realization of the documentary investigation of contents.

In the same way, a query was made in different databases, including Google Scholar, Scielo, Dialnet, Journal Citation Reports and Scimago. The search criteria consisted of: scientific articles in the area of educational mathematics, integral calculus and transcendental functions. The keywords used were the following: transcendental functions, approximative methods, school mathematical discourse and integral calculus. Based on the search for books and articles, an Excel database was created where approximately 40 books and more than 50 articles related to the teaching of mathematics, transcendental functions, their history and approximate methods were registered in an approximate time of two months.

Materials

For the presentation of the graphs and solved examples presented in this work, the Maple 2015 mathematical software was used. The computer used had a Linux Ubuntu version 18.04.5 LTS operating system and a processor Intel I7-7700@ 3600 GHz x 8.

Results

As part of the documentary research carried out in this work, a favorable trend was found for the pedagogy used to present the contents, mainly when a new edition is published. The main changes that are made are the use of a second printing ink (or more), more figures and diagrams, the incorporation of more demonstrative examples and a greater number of exercises proposed to the reader (Barnett, 1994; Edwards and Penney, 2007 ; Leithold, 2012; Purcell et al., 2007; Stewart, 2015). Similarly, some works include handling mathematical software such as Maple (Fox, 2011), MATLAB, GNU Octave (Lie, 2019), GeoGebra (Mora, 2018) and Excel (Torres, 2016), among others.

The integration form used in the calculation books is the same, which confirms the influence of the DME. In some texts, in their appendices, more integration formulas are attached, which have been obtained through the techniques of integration by parts, trigonometric, partial fractions, trigonometric processes; however, the essence of the form remains the same, since no new formulas are incorporated to integrate functions: the DME stands out again. As a result of the research carried out, it is proposed to add to the elementary form the formulas for the transcendental functions of Lambert W (x) (Corless, Gonnet, Hare, Jeffrey and Knuth, 1996) and Leal (Vazquez, Sandoval and Filobello, 2020). Table 1 presents the integration formulas under discussion.

Tabla 1. Nuevas fórmulas en cálculo integral

Función trascendente	Fórmula
Lambert $W(x)$	$\int W(x)dx = \frac{x(W^2(x) - W(x) + 1)}{W(x)} + C$
$\text{Lsinh}(x)$	$\int \text{Lsinh}(x)dx = -\text{Lsinh}(x)\cosh(\text{Lsinh}(x)) + \sinh(\text{Lsinh}(x)) + \text{Lsinh}(x)^2 \sinh(\text{Lsinh}(x)) + C$
$\text{Lcosh}(x)$	$\int \text{Lcosh}(x)dx = -\text{Lcosh}(x)\sinh(\text{Lcosh}(x)) + \cosh(\text{Lcoshh}(x)) + \text{Lcosh}(x)^2 \cosh(\text{Lcosh}(x)) + C$
$\text{Ltanh}(x)$	$\int \text{Ltanh}(x)dx = -\text{Ltanh}(\text{x})^2 \tanh(\text{Ltanh}(x)) - \left(\frac{\sum_{n=0}^{\infty} 2^{2n}(2^{2n}-1)B_{2n}(\text{Ltanh}(x))^{2n+1}}{(2n+1)!}\right) + C$
$\text{Lcosh}(x)$	$\int \text{Lcsch}(x)dx = -\text{Lcsch}(\text{x})^2 \csc(\text{Lcsch}(x)) - \left(\frac{\sum_{n=0}^{\infty} 2(-1)^{2n-1}(2^{2n-1}-1)B_{2n}(\text{Lcsch}(x))^{2n+1}}{(2n+1)!}\right) + C$
$\text{Lsech}(x)$	$\int \text{Lsech}(x)dx = \text{Lsech}(\text{x})^2 \,\text{sech}(\text{Lsech}(x)) - \left(\frac{\sum_{n=0}^{\infty} E_{2n}(\text{Lsech}(x))^{2n+2}}{(2n+2)(2n)!}\right) + C$
$\text{Lcoth}(x)$	$\int \text{Lcoth}(x)dx = -\text{Lcoth}(\text{x})^2 \coth(\text{Lcoth}(x)) - \left(\frac{\sum_{n=0}^{\infty} 2^{2n}(-1)^{2n}B_{2n}(\text{Lcoth}(x))^{2n+1}}{(2n+1)!}\right) + C$
$\text{Lln}(x)$	$\int \text{Lln}(x)dx = \frac{1}{2}(\text{Lln}(x)+1)^2\text{Lln}(\text{Lln}(x)+1) + \frac{1}{4}\text{Lln}(x)^2 - \frac{1}{2}\text{Lln}(x) + (-(\text{Lln}(x)+1)+1)\ln(\text{Lln}(x)+1) - \frac{3}{4} + C$
$\text{Ltan}(x)$	$\int \text{Ltanh}(x)dx = \text{Ltan}(\text{x})^2 \tan(\text{Ltan}(x)) - \left(\frac{\sum_{n=0}^{\infty} 2^{2n}(2^{2n}-1)\lvert B_{2n}\rvert(\text{Ltanh}(x))^{2n+1}}{(2n+1)!}\right) + C$

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$\mathrm{Lsinh}_2(x)$	$\int \mathrm{Lsinh}_2(x)dx = \frac{1}{2}\mathrm{Lsinh}_2(x)^2 + \mathrm{Lsinh}_2(x)\sinh\left(\mathrm{Lsinh}_2(x)\right) - \cosh\left(\mathrm{Lsinh}_2(x)\right) + C$
$\mathrm{Lcosh}_2(x)$	$\int \mathrm{Lcosh}_2(x)dx = \frac{1}{2}\mathrm{Lcosh}_2(x)^2 + \mathrm{Lcosh}_2(x)\cosh\left(\mathrm{Lcosh}_2(x)\right) - \sinh\left(\mathrm{Lcosh}_2(x)\right) + C$
Nota: *E* son los números de Euler *y B* son los números Bernoulli.	

Fuente: Vazquez *et al.* (2020)

The functions found in table 1 were recently published, except Lambert W, which has its history and in recent years its use has become very important in science and engineering. Leal functions also belong to the group of transcendental functions since they have new algebraic properties that allow clearances, and thus enrich algebra and trigonometry.

Another aspect to mention is about calculus books, since some of the topics presented remain practically the same, despite the fact that there are new theoretical approaches in relation to calculus. For example, the topic of applications of the definite integral includes the calculation of areas, solids of revolution, arc length, among others.

The integration techniques that always appear in calculus books by well-known authors are presented in Table 2. In this same table, other integration methods that are little known but are considered relevant have been added.

Tabla 2. Las diferentes técnicas de integración

Método	Descripción de la técnica de integración en libros conocidos
Funciones elementales	Se aplican paso a paso las reglas básicas de integración como la de la suma y la regla de la potencia. Generalmente, los integrandos son polinomios con exponentes enteros o racionales (Astey, 2009; Ayres y Mendelson, 1994; Edwards y Penney, 2007; Garza, 1990, 2017; Leithold, 2012; Lizama, 2005; Aguilar, Bravo, Gallegos, Cerón y Reyes, 2010; Purcell *et al*., 2007; Stewart, 2012, 2015; Swokowski, 1989; Thomas, Finney, Weir y Giordano, 2003; Zill, 1987).
Por partes	*No* hay un esquema didáctico definido. Por inspección y por experiencia se selecciona u y dv en los integrandos (mismas referencias).
Trigonométrica	Existe una guía de integración para la combinación de seno-coseno, secante-tangente, cotangente-cosecante. Los productos seno y coseno pueden tener diferentes argumentos. Se utilizan las identidades trigonométricas (mismas referencias).
Fracciones parciales	Son cuatro casos. Factores lineales, factores lineales repetidos, factores cuadráticos distintos, factores cuadráticos. Generalmente, se utiliza la técnica de coeficientes indeterminados para encontrar los coeficientes en los numeradores (mismas referencias).
Sustitución trigonométrica	Se presenta una guía de integración cuando hay radicales o cocientes de funciones trigonométricas. El cambio de variable consiste en reemplazar las expresiones algebraicas por expresiones trigonométricas o una expresión trigonométrica por un cociente de expresiones algebraicas, según sea el caso. Se emplea el triángulo rectángulo para los cambios de variable (mismas referencias).
Hermite-Ostrogradsky	Utilizado para expresiones racionales cuando en el denominador hay factores lineales o cuadráticos con multiplicidad. La fracción del integrando se descompone en la derivada de cociente, donde el denominador es el producto de los factores disminuidos en 1 sus respectivas multiplicidades. El numerador es un polinomio en 1 orden menor al exponente del denominador. A esta fracción se le suma el número de integrales para cada uno de los

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	factores. Se trata de integrales fáciles de resolver (Del Hoyo y Muto, 2001; Demidovich y Aparicio, 2001)
Alemán	Para integrales con cocientes de funciones irracionales. El proceso consiste en determinar la derivada del producto de un polinomio de grado $n-1$ en relación con el numerador del integrado por la raíz del denominador del integrado (Demidovich y Aparicio, 2001).
Euler	Para integrales con cocientes de funciones irracionales. Existen tres soluciones dependiendo de los coeficientes de la raíz del trinomio de segundo grado que se encuentra en el denominador (Del Hoyo y Muto, 2001; Demidovich y Aparicio, 2001)
Chebyshev	Se aplica a integrales binomias, que consisten en el producto de x elevado a una potencia con un binomio elevado a una potencia. Hay tres casos que permiten la solución en términos de funciones elementales. En todos ellos se realiza una operación entre los exponentes y el resultado debe ser un número entero y dependiendo del caso será el tipo de cambio de variable empleado (Demidovich y Aparicio, 2001).

Fuente: Elaboración propia

In table 2 it can be seen that the integration techniques used in classic calculus books are: fundamental integration, integration by substitution of change of variables, integration by parts, trigonometric integration (circular and hyperbolic), integration by partial fractions and integration by other trigonometric substitution techniques (methods using the right triangle); the other integration techniques do not even see its existence. However, part of DME in the classroom is turning to traditional books. Del Hoyo and Muto (2001) present the Hermite-Ostrogradsky and Euler methods; on the other hand, Demidovich and Aparicio (2001) follow the DME and the aspect of letting the student test his intuitive abilities by using the formulas of the Chebyshev and German methods.

In that sense, the books provide a guide to solve problems through the usual integration techniques, which are handled in integral calculus courses. In the case of integration by parts, an intuitive solution by inspection strategy is usually presented focusing on the “arbitrary” choice of u and dv, according to the integration by parts formula (1).

$$\int udv = uv - \int vdu. \quad (1)$$

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ISSN 2007 - 7467

Therefore, considering the integration schemes, we propose the following didactic solution guide to integrate by parts, avoiding the DME imposed in this integration technique both in books and in the classroom (see figure 1).

Figura 1. Guía didáctica de solución para integrar por partes

Caso 1. La integral del producto de una función trascendente por una función algebraica

$$\int x^n f(x)dx$$

donde $u = x^n, dv = f(x)dx$. En este caso $f(x)$ puede ser una función trascendente excepto logaritmo y tangente.

Caso 2. La integral del producto de una función trascendente por una función algebraica

$$\int f(x)^{-1} x^n dx$$

donde $u = f(x)^{-1}, dv = x^n$. En este caso $f(x)^{-1}$ puede ser una función trascendente de arco, logaritmo, excepto la función exponencial.

Fuente: Elaboración propia

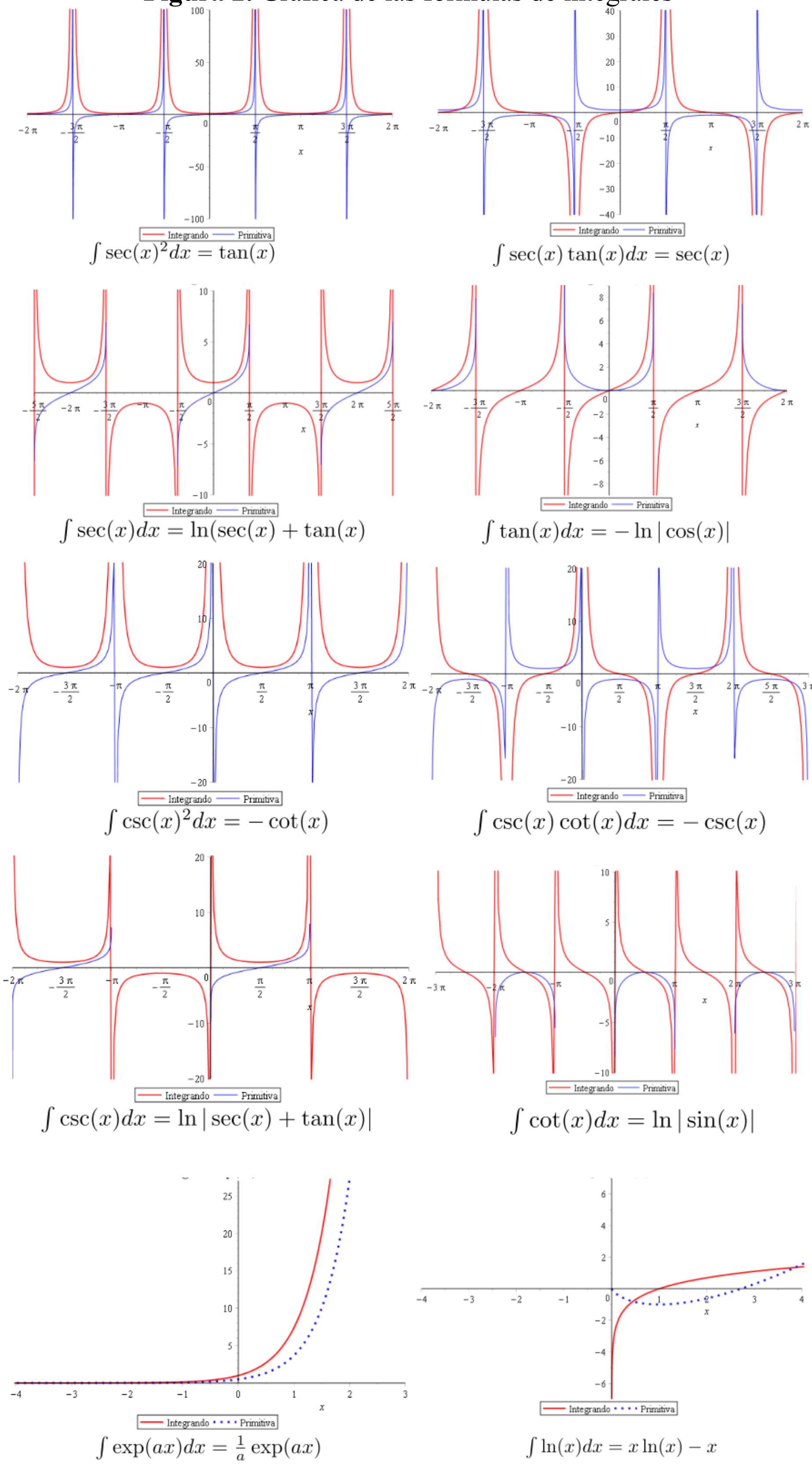
Note that the integration guide is divided into two cases, which consist, first, in identifying the transcendental functions and their inverse functions. Thus, in case 1 if $f(x) = \sin x$, then it is selected as dv to integrate. It is important to note that the case of the tangent function has been excluded since an integral by parts with this function cannot be integrated and obtain a result in terms of elementary functions. For case 2 we have $u = f(x)^{-1}$ excludes the exponential function and this is because the logarithm function meets the requirement and is the inverse function of the exponential function.

Discussion

The integral calculation form includes the formulas to integrate elementary functions, however, no graph of the functions obtained when integrated is presented. It is assumed that the student has a solid foundation in managing functions, an assumption made from the DME. Figure 2 shows the graphs of some of the functions obtained with integration formulas most used in integral calculus courses. The periodicity of the primitive functions that have been obtained by integrating the transcendental functions can be observed. It is important to note that the integral of a periodic function returns another periodic function. The DME that prevails in books and in integral calculus classes makes this detail obvious and assumes that every student has the necessary pre-calculus knowledge. In this research work, it is proposed to the authors of books and teachers to analyze the functions that are obtained in the form and, additionally, to graph the primitive function that is obtained in the exercises proposed to be carried out.

In addition, the behavior of the integrals for the natural exponential function and natural logarithm is shown. See, in the case of the exponential function, the effect of the constant that appears next to the integration variable: it generates a scaling in the function. In the case of the integral of the logarithm function, the resulting function is different since algebraic terms are generated that generate a change in the function. Note also that the discontinuity at the origin has disappeared, so the primitive function is now analytic at the origin.

Figura 2. Gráfica de las fórmulas de integrales



Fuente: Elaboración propia

When studying the topic of integration by parts, it is common for students to have doubts when choosing u and dv in an integrand. Textbooks are only limited to explaining the problem and making the respective selection of the integrand. For example, Ayres and Mendelson (1994), Edwards and Penney (2007), Garza (2017), Leithold (2012), Lizama (2005), Purcell et al. (2007), Stewart (2015), Swokowski (1989) and Zill (1987) present demonstrative examples in the solution of integral problems by parts making the selection of u and dv by inspection. Although pedagogically they do not explain in detail why, the experience obtained by intuition or by practice when solving a stack of problems shows us that the trick is to look for algebraic cancellations, which is generally caused by the presence of quotients in the integrand. , thus obtaining a simplified expression that generates a simpler or more direct integral to solve.

Practice tells us that the selection of u and dv has a hierarchy in its selection. To select u correctly, the inverse functions of the trigonometric ones must first be chosen, then the logarithmic, algebraic, trigonometric and finally the exponential ones. Teachers in classrooms often call this rule the rule of the Alps, because the selection priority is: arc functions, logarithms, algebraic expressions raised to a power, exponentials and, finally, the trigonometric functions sine and cosine. Popular knowledge has also nicknamed the general formula for solving second degree equations as the pork rind.

In this regard, Mateus (2016) carried out a didactic analysis of the method of integration by parts in an undergraduate class. Some errors that occur in this integration technique are reported here as the selection of u and dv is not clear. Likewise, the same author emphasizes that according to the Ontosemiotic approach to Mathematical Cognition and Instruction, there is a semiotic (epistemic) conflict since the teacher avoids the hierarchy by choosing a from u and dv. Another aspect reported by the author is the cognitive semiotic conflict where the teacher presents the students with the integral of the quotient of a sine function and an exponential function. Faced with the impossibility of some of the students in solving it, the teacher intervenes indicating that if the exponential is raised, it can already be integrated.

The cognitive conflicts experienced by undergraduate students in the instruction of the topic of integration by parts reported by Mateus (2016) also occur at the upper secondary level in the Integral Calculus subject. Seen in the light of the DME, the act of selecting u and dv by inspection, or in its case dictating to students as a recipe that the selection of u and dv has a hierarchy without giving any explanation of why, is to perpetuate symbolic violence, since arguments and procedures are being imposed (Soto and Cantoral, 2014). As a result, the student remains unclear on how to choose the elements of the integrand.

Taking into account the hierarchy when selecting u and dv, we obtain the didactic solution guide for the solution to integrate by parts, which is proposed in this work. In this way, it is much easier to identify the factors that need to be derived and integrated when using (1). It is the responsibility of the teacher to clarify, according to their teaching strategies, the hierarchy of selection of u and dv and to show the student that the purpose of choosing u accurately is to generate simplifications through division, or in his case cancel algebraic factors by decreasing the exponents to which they are raised.

On the other hand, in integral calculus textbooks, and especially at the high school level, when integration by partial fractions is approached, the DME limits the student to believing that the methodology based on the four cases of decomposition and the method of the undetermined coefficients to obtain the constants of the numerators of this decomposition are the only way to do it; However, this is not so. Fortunately, some authors challenge the DME by employing other solution techniques to obtain the constants of the residual decomposition method. Ogata (2003) and D'Azzo and Houpis (1995) use this method to determine the inverse Laplace transform, which gives solutions to differential equations that are the mathematical model of different systems, such as the inverted pendulum and the mass system. -spring-damper, etc. Gómez (2017) presents the methodology in a didactic way to obtain the expansion by partial fractions through residuals with illustrative examples, however, fraction expansions are still presented using real numbers in the denominators, as suggested by the DME. In the work of Ambardar (1999) analog and digital signal processing, the design of digital filters and some of their applications are studied. In digital signal processing, a complex variable is used to perform the different analyses. In this book the author performs a partial fraction expansion using complex numbers in the denominators to compute the inverse z-transform. However, the calculus books say nothing about it. In the case of pre-calculus books, complex numbers, algebra and trigonometry are addressed, among other topics, and despite exposing complex numbers, complex numbers are not used in the expansion of partial fractions (Barnett, 1994; Barnett, Ziegler and Byleen, 2012). Likewise, it is also possible to use complex numbers in the expansion of partial fractions to obtain the inverse Laplace transform in differential equations courses. (D'Azzo y Houpis, 1995; Ogata, 2003).

Below are the alternate procedures not considered in typical integral calculus courses and that can be implemented both in calculus books and in school courses. In the first five examples of integration by parts are presented using the didactic solution guide proposed in this article. Number five presents an example of integration by partial fractions that uses complex numbers in its solution.

Alternative solution procedures

Example 1

For case 1, let the integral be

$$\int x\cos x\,dx. \tag{2}$$

The integrate consists of a product of a linear algebraic function and a transcendental function such that we have $\int x^n f(x)dx$. Therefore, using the left-hand side of (1) we get $u = x$, con $n = 1$, $dv = \cos x\,dx$. Consequently, we get the following:

$$du = dx,$$

$$v = \sin x. \tag{3}$$

Substituting (3) on the right hand side of the given formula for (1), the primitive function we are looking for is

$$\int x\cos x\,dx. = \cos x + x\sin x + C. \tag{4}$$

Example 2

For case 2, let the integral be:

$$\int \ln x\,dx. \tag{5}$$

In this example we have an integrand that contains a function of the type $f\,(x)^{-1}$ given by the function $\ln x$ (remember that the logarithm function is the inverse of the exponential function). Therefore, the integral given by (5) is of case 2 given by $\int f(x)^{-1}x^n dx$. Considering (1) we have $u = f(x)^{-1} = \ln x\,, dv = dx$ con $n = 0$. Consequently, using the integration by parts formula, we obtain

$$du = \frac{1}{x}dx,$$

$$v = x. \tag{6}$$

By using the integration by parts formula given by (1), we obtain the following:

$$\int \ln\ x\ dx = xln\, x - x + C. \tag{7}$$

Example 3

It is possible to solve integrals that have in their integrated the product of two algebraic factors. Let the integral be:

$$\int x\sqrt{1+x}dx. \tag{8}$$

This integral corresponds to the case we have $\int x^n f(x)dx$. Making $u = x$ n= 1, $dv = \sqrt{1+x}$ we have:

$$du = 1,$$

$$v = \frac{2}{3}(1+x)^{\frac{3}{2}}. \tag{9}$$

And using (1) we find the solution of the integral (8):

$$\int x\sqrt{1+x}dx = \frac{2}{3}x(1+x)^{\frac{3}{2}} - \frac{4}{15}(1+x)^{\frac{3}{2}} + C. \tag{10}$$

Example 4

Solve the following integral:

$$\int xW(x)dx, \tag{11}$$

There, $W(x)$ is the Lambert W function. Following Corless et al. (1996) and Vazquez, Sandoval, Garcia, Herrera and Filobello (2019), Lambert W is defined as follows:

$$x = W(x)\exp W(w). \tag{12}$$

To solve this integral by parts, we must resort to the change of variable.

$$t = W. \tag{13}$$

We are going to integrate (13) using the first formula of the table form and using case 1 for integration by parts and not incurring the DME. In this case:

$$u = x, n = 1, dv = dx,$$

$$du = dx, v = \frac{x^2(W^2-W+1)}{W}. \tag{14}$$

Using (1) we have:

$$\int xW(x)dx = x\frac{x(W^2-W+1)}{W} - \int \frac{x(W^2-W+1)}{W}dx\,. \tag{15}$$

After substituting (13) in (15) and completing the differential with $dx = t\exp t + \exp t$, is obtained:

$$\int xW(x)dx = x\frac{x(W^2 - W + 1)}{W} - \int \big(\exp 2t\ (1+t)\big)\, t^2 - \big(\exp 2t\ (1+t)\big)t + \big(\exp 2t\ (1+t)\big)\,)dx. \quad (16)$$

From the resulting integral, it must be integrated again by parts taking into account case 1. Once we proceed, we obtain:

$$\int xW(x)dx = x\frac{x(W^2-W+1)}{W} - \left[\frac{\exp\ 2t}{8}(4t^3 - 6t^2 + 6t + 1)\right]. \quad (17)$$

To retrieve W we do $\exp 2t = \frac{x^2}{W^2}, t = w$. After substituting in (17) and simplifying we obtain:

$$\int xW(x)dx = \frac{x^2(4W^3-2W^2+2W-1)}{8W^2}. \quad (18)$$

Example 5

Solve the integral by partial fractions:

$$\int \frac{8dx}{x(x^2+4)}. \quad (19)$$

The integral that we will solve generates two fractions in its decomposition. According to the DME that we find in textbooks, the expansion would correspond to using the case 1 distinct linear factors and the case 3 distinct quadratic factors. Therefore, the expansion that is carried out in the usual way would be:

$$\frac{8}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}, \quad (20)$$

In this case, the constants A, B and C can be determined using the methodology of undetermined coefficients or by the method of residuals. We will do the expansion of partial fractions considering only case 1, that is, different linear factors. In this way, you get:

$$\frac{8}{x(x^2+4)} = \frac{A}{x} + \frac{B}{x-2i} + \frac{C}{x+2i}. \quad (21)$$

Using any solution method to find the constants A,B,C we then have

$$\frac{8}{x(x^2+4)} = \frac{2}{x} - \frac{1}{x-2i} - \frac{1}{x+2i}. \quad (22)$$

The integral that we will solve is based on three simple fractions that are easy to solve:

$$\int \frac{8dx}{x(x^2+4)} = \int \frac{2dx}{x} - \int \frac{dx}{x-2i} + \int \frac{dx}{x+2i}. \quad (23)$$

Of (22) and (23) We clearly observe that in the decomposition complex numbers have remained and that when performing the integrals, imaginary terms remain within the arguments of the logarithms generated in the second and third integrals. Applying the properties of logarithms and simplifying, we obtain:

$$\int \frac{8dx}{x(x^2+4)} = 2\ln x - \ln(x^2+4) + C. \qquad (24)$$

The advantage of using complex numbers is that in this example the use of integration by substitution by change of variable was avoided, since integrals two and three that were solved were much simpler because it was direct integration. In the final result it is possible to observe that real terms remained, as would have happened when following the traditional procedures in the expansion of the partial fractions, a procedure influenced by the DME.

Other integration techniques for rational and irrational functions

Below we will show the Hermite-Ostrogradsky and German methods presented in Del Hoyo and Muto (2001) and Demidovich y Aparicio (2001).

Hermite–Ostrogradsky method

We can consider this method an extension of integration by partial fractions with the advantage that at the end of the procedure a simple integral is solved for each of the different factors, which can be linear or quadratic, however, the inconvenience of solve tedious derivatives. The method is presented in Figure 3.

Figura 3. Método Hermite-Ostrogradsky

$$\frac{R(x)}{F(x)} = \frac{d}{dx}\left(\frac{\psi(x)}{\phi(x)}\right) + \frac{A}{x-a} + \frac{B}{x-b} + \cdots + \frac{Mx+N}{(x-\gamma)^2+\beta} + \cdots. \quad (25)$$

El denominador $\phi(x)$ consiste en la multiplicación de factores con una unidad menos en la multiplicidad. El numerador es un polinomio general de un grado inferior a $\phi(x)$. Una vez realizada la descomposición se tiene

$$\int \frac{R(x)}{F(x)}dx = \int \frac{d}{dx}\left(\frac{\psi(x)}{\phi(x)}\right)dx + \int \frac{Adx}{x-a} + \int \frac{Bdx}{x-b} + \cdots. \quad (26)$$

Por último se determinan los coeficientes de los numeradores.

$$\int \frac{R(x)}{F(x)}dx = \frac{\psi(x)}{\phi(x)} + A\ln(x-a) + B\ln(x-b) + \cdots. \quad (27)$$

Fuente: Elaboración propia

Example 6

Determine the integral using the Hermite-Ostrogradsky method:

$$\int \frac{x^2-4}{x^3(x^2+1)^2}dx. \quad (28)$$

In this case, the expansion for this integral will be as follows:

$$\frac{x^2-4}{x^3(x^2+1)^2} = \frac{d}{dx}\left[\frac{Ax^3+Bx^2+Cx+D}{x^2(x^2+1)}\right] + \frac{E}{x} + \frac{Fx+G}{x^2+1}. \quad (29)$$

Differentiating and simplifying the derivative, we obtain:

$$\frac{x^2-4}{x^3(x^2+1)^2} = -\frac{Ax^5+2Bx^4-Ax^3+3Cx^3+4Dx^2+Cx+2D}{x^3(x^2+1)^2} + \frac{E}{x} + \frac{Fx+G}{x^2+1}. \quad (30)$$

To determine the constants in (30) we proceed to form the system of linear equations equaling coefficients for each of the powers.

$$\begin{aligned}
x^0 &: \quad -2D = -4,\\
x^1 &: \quad -C = 0,\\
x^2 &: \quad -4D + E = 1,\\
x^3 &: \quad A - 3C + G = 0,\\
x^4 &: \quad -2B + 2E + F = 0,\\
x^5 &: \quad -A + G = 0,\\
x^6 &: \quad E + F = 0.
\end{aligned} \quad (31)$$

Once (31) is solved with any algebraic solution method, we get $A = 0, B = \frac{9}{2}, C = 0, D = 2, E = 9, F = -9, G = 0$. Thus, we substitute in our integral:

$$\int \frac{x^2-4}{x^3(x^2+1)^2}dx = \frac{\frac{9}{2}x^2+2}{x^2(x^2+1)} + \int \frac{9}{x}dx + \int \frac{-9x}{x^2+1}dx\,. \tag{32}$$

Finally, integrating through (27) we arrive at:

$$\int \frac{x^2-4}{x^3(x^2+1)^2}dx = \frac{\frac{9}{2}x^2+2}{x^2(x^2+1)} + 9\ln x - \frac{9}{2}\ln(x^2+1)\,. \tag{33}$$

Method (German) for integrals of irrational functions

This method can be considered as an extension of the change of variable trigonometric substitution method with the advantage that the procedures are simpler with a resulting integral that is easy to solve. This integration technique is presented in Figure 4 (Demidovich y Aparicio, 2001).

Figura 4. Método (alemán) para integrales de funciones irracionales

$$\int \frac{P_n(x)dx}{\sqrt{ax^2+bx+c}} = Q_{n-1}(x)\sqrt{ax^2+bx+c} + \int \frac{Kdx}{\sqrt{ax^2+bx+c,}} \tag{34}$$

Derivando

$$\frac{P_n(x)dx}{\sqrt{ax^2+bx+c}} = \frac{d}{dx}\left(Q_{n-1}(x)\sqrt{ax^2+bx+c}\right) + \frac{Kdx}{\sqrt{ax^2+bx+c,}} \tag{35}$$

Fuente: Elaboración propia

$P_n(x)$ is a polynomial of degree n, $Q_{n-1}(x)$ is a polynomial of degree $n-1$ with coefficients to be determined and K it is a constant. Note that the resulting integral in (23) will be completed using the formulas that appear in the integral calculation forms.

Example 7

Solve the integral:

$$\int \frac{(6x^3+6x)\,dx}{\sqrt{x^2+9}}, \tag{36}$$

We start by substituting in (35):

$$\frac{(6x^3+6x)}{\sqrt{x^2+9}} = \frac{d}{dx}\left[(Ax^2+Bx+C)\sqrt{x^2+9}\right] + \frac{K}{\sqrt{x^2+9}}. \qquad (37)$$

Doing the derivative and simplifying:

$$\frac{(6x^3+6x)}{\sqrt{x^2+9}} = \frac{3Ax^3+2Bx^2+18Ax+Cx+9B+K}{\sqrt{x^2+9}}. \qquad (38)$$

To determine the constants in (38), we proceed to form the system of linear equations equaling coefficients for each of the powers.

$$\begin{aligned} x^0&: 9B+K=0,\\ x^1&: 18A+C=6,\\ x^2&: \quad 2B=0,\\ x^3&: \quad 3A=6. \end{aligned} \qquad (39)$$

Solving (39) with any algebraic solution method we obtain $A=2, B=0, C=-30, K=0$. Substituting in (23) we have

$$\int \frac{(6x^3+6x)\,dx}{\sqrt{x^2+9}} = \frac{2x^2-30}{\sqrt{x^2+9}}. \qquad (40)$$

From (40) we can see the advantage of using this integration method because the procedure was fast and completely algebraic. In this example the constant of the integral was equal to zero and thus it was not necessary to carry it out. Comparing this integration technique against the change of variable trigonometric substitution method, it would have been necessary to make use of trigonometric identities, the use of the right triangle, both to obtain the new variable in the trigonometric domain and to recover the variable in x with the possibility of performing a spectacular algebraic simplification, including the integration process that may require the use of a second integration technique such as integration by parts.

Conclusions

In this article it was proposed to extend the basic form of integral calculus with the Lambert W function and other transcendental functions that have new algebraic properties. In addition, a didactic solution strategy was proposed to solve integrals by parts. Some of the graphs of the functions obtained when integrating functions of the basic integration form were presented. In this research work, four problems were solved using the proposed methodology for integration by parts.

In the integration by partial fractions it was shown with an example that it is possible to simplify algebraic processes by using complex numbers. In the same way, two integration examples have been presented, which were solved using the little-spread integration techniques, Hermite-Ostrogradsky and the method for integrals of irrational functions. These integration methods showed the advantage of being algebraic and avoiding complicated integrals.

From the solved exercises it is concluded that it is possible to show the student different alternatives to solve integrals through additional techniques, with some procedural advantages compared to those commonly taught in integral calculus courses.

The DME has limited the student's thinking by boxing it only in the conventional integration techniques that are published in the best known calculus books that the DME itself imposes on the courses. Possibly, because these books have become a tradition in public and private institutions, high schools or universities for their integral calculus courses. We believe that these alternative procedures presented in this work for integral calculus can be incorporated into textbooks in future editions and in their teaching.

Future lines of research

It is necessary to investigate educational applications for the new integrals in table 1, except Lambert W, which currently has wide applications in the different branches of technology and science. Pilot groups should be formed to put into practice what is proposed in this work at the upper middle and higher levels. Subsequently, it is essential to design and apply evaluation instruments that allow verifying the student's learning regarding the didactic solution guide to integrate by parts.

Limitations

Due to the large number of topics that are covered in a semester course on integral calculus, time would be a limitation to implement all the integration techniques, since they are completely procedural. In addition, to implement the integration formulas of the Lambert W and Leal functions, teachers are required to update their knowledge base. In the case of high school students and some undergraduate students, they do not know the Euler and

Bernoulli numbers that are handled in the formulas of the family of Leal functions, so it is necessary for teachers to retake these concepts.

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Revista Iberoamericana para la Investigación y el Desarrollo Educativo
ISSN 2007 - 7467

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